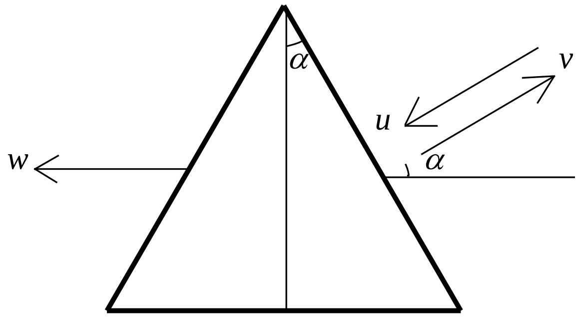
9 (i)



By *CLM* of particle parallel to the inclined plane surface of the cone, $v \parallel u$ **E1**

CLM horizontally for system **M1** $m\,u\cos\alpha = M\,w - m\,v\cos\alpha$ **A1**

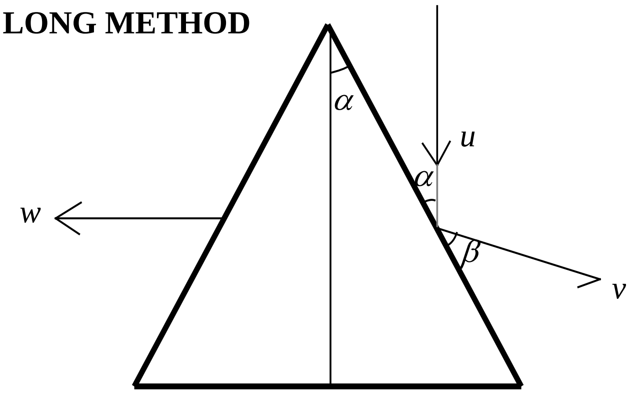
NEL perp^r. to plane of contact **M1** $e\,u = v + w\cos\alpha$ **A1**

M1 for subst^g. back and a good attempt at isolating *w*

Thus $m\,u\cos\alpha = M\,w - m(e\,u - w\cos\alpha)\cos\alpha \Rightarrow w = \dfrac{mu(1+e)\cos\alpha}{M + m\cos^2\alpha}$ **A1**

7

(ii) **LONG METHOD**



CLM horizontally for system **M1** $M\,w = m\,v\sin(\alpha+\beta)$ **A1**

NEL perp^r. to plane of contact **M1** $e\,u\sin\alpha = v\sin\beta + w\cos\alpha$ **A1**

CLM for *P* $\parallel$ to slope **M1** $m\,u\cos\alpha = m\,v\cos\beta$ **A1**

M1 for use of $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and eliminating β

$v\cos\beta = u\cos\alpha$ and $v\sin\beta = e\,u\sin\alpha - w\cos\alpha$

to get $w = \dfrac{mu(1+e)\sin\alpha\cos\alpha}{M + m\cos^2\alpha}$ **A1**

SHORT METHOD

Component of particle's velocity parallel to slope does not affect the motion of the cone, so w is as before but with u replaced by $u \sin \alpha$. **M7 A1**

8

$$w = k.\frac{\sin 2\alpha}{M + m\cos^2\alpha}$$

$$\Rightarrow \frac{\mathrm{d}w}{\mathrm{d}\alpha} = k.\frac{\left(M + m\cos^2\alpha\right)2\cos 2\alpha - \sin 2\alpha\left(-2m\sin\alpha\cos\alpha\right)}{\left(M + m\cos^2\alpha\right)^2}$$

M1 for diff^n. attempt using the quotient rule

dM1 for equating numerator to zero

B1 for using appropriate trig. identities to get all in terms of cos α

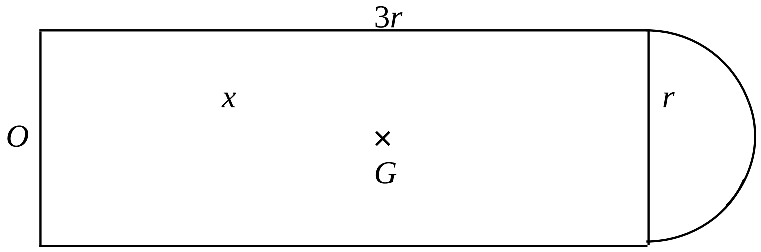
$(M + m\,c^2)(2\,c^2 - 1) + 2\,m\,c^2\,(1 - c^2) = 0$ **A1 correct to here**

$\Rightarrow\ 0 = 2M\,c^2 + 2m\,c^4 - M - m\,c^2 + 2m\,c^2 - 2m\,c^4$

$\Rightarrow\ M = (2M + m)\,c^2$ and $\cos\alpha = \sqrt{\dfrac{M}{2M + m}}$ **A1 (GIVEN ANSWER)**

5

10 ---



Let $OG = x$.

M1 for finding position of C. of G. **dM1 for $(\Sigma m_i)\, x = \Sigma(m_i\, x_i)$**

$$\left(\pi r^2.3r.\rho + \frac{2}{3}\pi r^3.3\rho\right)x = 3\pi r^3\rho.\frac{3r}{2} + 2\pi r^3\rho\left(3r + \frac{3r}{8}\right)$$

B1 **B1** **B1** **B1**

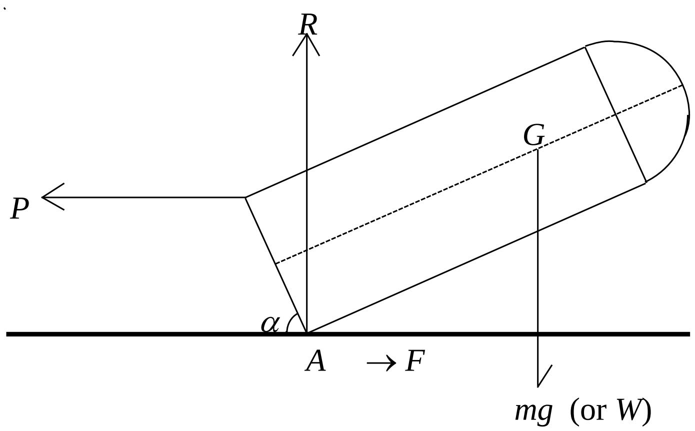
Mass of figure C of G of solid hemisphere

[N.B. May include g's throughout or have cancelled ρ's automatically.]

Dividing by $\pi r^3 \rho \;\Rightarrow\; 5x = \frac{9r}{2} + 6r + \frac{3r}{4} \;\Rightarrow\; x = \frac{9r}{4}$ **M1 A1**

Must be correct distance from their point of reference

8



Assuming no tilting, $R = mg$, $P = F$ and $F = \mu R \Rightarrow P = \mu mg$ **M1 A1**

Assuming no sliding, $\underline{A\circlearrowleft}$ $P\,.\,2r\sin\alpha = mg\left(\frac{9r}{4}\sin\alpha - r\cos\alpha\right)$

M1 **B1** **dM1 A1**

$$\Rightarrow P = mg\left(\frac{9}{8} - \frac{1}{2}.\frac{\cos\alpha}{\sin\alpha}\right)$$ **A1**

and figure tilts before it slides provided $\frac{9}{8} - \frac{1}{2}\cot\alpha < \mu$ **B1 for correct conclusion**

8

M1 for considering *P* in other direction

$R = mg, \quad P = F$ and $F = \mu R \Rightarrow P = \mu mg$ with G to the left of A **A1**

$\underline{A \circlearrowleft}$ $\quad P \,.\, 2r \sin \alpha = mg \left(r \cos \alpha - \frac{9r}{4} \sin \alpha \right)$ **A1**

leading to $\mu > \frac{1}{2} \cot \alpha - \frac{9}{8}$ **A1**

4

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11 (i) N.B. $\tan\theta = \frac{1}{2} \Rightarrow \sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$ **B1**

$x = v\,t\cos\theta = 10\,t\sqrt{5}$ **B1**

so $\underline{\mathbf{i}}$ – component is $50 -$ their $x \times \cos 60^{\circ}$ **M1** $= 50 - 5\,t\sqrt{5}$ **A1**

and (the M1 is for either of these)

$\underline{\mathbf{j}}$ – component is their $x \times \sin 60^{\circ}$ $= 5\,t\sqrt{15}$ **A1**

$y = v\,t\sin\theta - \frac{1}{2}g\,t^2$ **M1** $= 5\,t\sqrt{5} - 5\,t^2$ **A1**

i.e. $\underline{\mathbf{r}} = \left(50 - 5t\sqrt{5}\right)\mathbf{i} + \left(5t\sqrt{15}\right)\mathbf{j} + \left(5t\sqrt{5} - 5t^2\right)\mathbf{k}$

7

Then $OP = 5\sqrt{\left(10 - t\sqrt{5}\right)^2 + \left(t\sqrt{15}\right)^2 + \left(t\sqrt{5} - t^2\right)^2}$

M1 for attempt at this with decent squaring attempt

$= 5\sqrt{100 - 20t\sqrt{5} + 5t^2 + 15t^2 + 5t^2 - 2t^3\sqrt{5} + t^4}$

$= 5\sqrt{t^4 - 2\sqrt{5}\,t^3 + 25t^2 - 20t\sqrt{5} + 100}$

$= 5\left(t^2 - t\sqrt{5} + 10\right)$ **A1 from fully correct working (GIVEN ANSWER)**

2

M1 for diff^{g}. or completing the square

$OP = 5\left(\left[t - \frac{1}{2}\sqrt{5}\right]^2 - \frac{5}{4} + 10\right)$

$\Rightarrow OP_{\min}$ when $t = \frac{1}{2}\sqrt{5}$ **dM1 for finding the time at which *OP* is minimised**

Then $\underline{\mathbf{p}} = \frac{\mathbf{75}}{\mathbf{2}}\mathbf{i} + \frac{25\sqrt{3}}{2}\mathbf{j} + \frac{25}{4}\mathbf{k}$ **B1 ft**

Horizontal bearing from O is then $\tan^{-1}\left(\frac{\mathbf{i}}{\mathbf{j}}\right) = \tan^{-1}\sqrt{3} = (0)60^{\circ}$ **A1 cao**

4

(ii) **M1 for diff^{g}. their k – component (= $5\,t\sqrt{5} - 5\,t^2$) from earlier (or equivalent)**

A1 for showing that $t = \frac{1}{2}\sqrt{5}$ **here also**

2

(iii) When $t = \tfrac{1}{2}\sqrt{5}$, $OP = \dfrac{175}{4}$ or $43\tfrac{3}{4}$ **M1 for finding $OP_{\min}$**

dM1 for finding time taken for bullet to reach *P*: $\dfrac{175/4}{350} = \dfrac{1}{8}$ sec.

M1 for attempt at speed of particle at this time:

$$\left|\left(-5\sqrt{5}\right)\mathbf{i} + \left(5\sqrt{15}\right)\mathbf{j} + (0)\mathbf{k}\right| = 5\sqrt{5}\sqrt{1^2 + 3^2} = 10\sqrt{5}$$

M1 for finding distance moved by particle in this time:

$10\sqrt{5} \times \dfrac{1}{8} \approx \dfrac{22}{8} \approx 3$ m **A1 (GIVEN ANSWER) cao from sensible approx$^{\text{n}}$. work**

OR

(iii) When $t = \tfrac{1}{2}\sqrt{5}$, $\underline{\mathbf{r}}_0 = \dfrac{75}{2}\mathbf{i} + \dfrac{25\sqrt{3}}{2}\mathbf{j} + \dfrac{25}{4}\mathbf{k}$ and $OP = \dfrac{175}{4}$ or $43\tfrac{3}{4}$

M1 for finding $OP_{\min}$ and $\underline{\mathbf{r}}_0$

dM1 for finding time taken for bullet to reach *P*: $\dfrac{175/4}{350} = \dfrac{1}{8}$ sec.

When $t = \tfrac{1}{2}\sqrt{5} + \dfrac{1}{8}$, $\underline{\mathbf{r}}_1 = \left(\dfrac{75}{2} - \dfrac{5\sqrt{5}}{8}\right)\mathbf{i} + \left(\dfrac{25\sqrt{3}}{2} + \dfrac{5\sqrt{15}}{8}\right)\mathbf{j} + \left(\dfrac{25}{4} - \dfrac{5}{64}\right)\mathbf{k}$

M1 for finding $\underline{\mathbf{r}}_1$

M1 for finding difference and its magnitude

$$\underline{\mathbf{r}}_{\text{diff}} = \left(\frac{5\sqrt{5}}{8}\right)\mathbf{i} + \left(\frac{5\sqrt{15}}{8}\right)\mathbf{j} - \frac{5}{64}\mathbf{k} = \frac{5}{64}\left(8\sqrt{5}\,\mathbf{i} + 8\sqrt{15}\,\mathbf{j} - \mathbf{k}\right)$$

and

$$\left|\mathbf{r}_{\text{diff}}\right| = \frac{5}{64}\sqrt{320 + 960 + 1} = \frac{5}{64}\sqrt{1281} \approx \frac{5}{64} \times 36 = 2\tfrac{13}{16} \approx 3$$

A1 (GIVEN ANSWER) cao from sensible approx$^{\text{n}}$. work

5

12 (i) p(one die gives at least one 6 in first r throws) = 1 – p(die gives no 6s in first r throws)

$= 1 - q^r$ **M2 A1**

OR B1 for $p + qp + q^2p + q^3p + \ldots + q^{r-1}p = \dfrac{p(1-q^r)}{1-q} = 1 - q^r$ **M1 A1**

Then p(both dice have given 6 at the r^{th} throw) **M1**

= p(both dice give 6 in first r throws) – p(both dice give 6 in first $r-1$ throws)

$= (1-q^r)^2 - (1-q^{r-1})^2$ **B1 for use of independence of events**

$= (1-q^r-1+q^{r-1})(1-q^r+1-q^{r-1})$ by the difference of two squares

$= q^{r-1}(1-q)\,.\,(2-q^{r-1}-q^r)$

$= p\;q^{r-1}\;(2-q^{r-1}-q^r)$ **A1 (GIVEN ANSWER)**

OR

M2 for correct approach

P_r = p(neither die gives 6 in first $r-1$ throws, then both give 6 on r^{th} throw)

+ p(one die gives a 6 before the r^{th} throw, then 2^{nd} die first gives a 6 on r^{th} throw)

$= (q^2)^{r-1}.p^2 + 2\times(1-q^{r-1})\times(q^{r-1}p)$ **B1 for use of independence of events**

A1 A1 for correct, unsimplified probs.

$= (pq^{r-1})[(1-q)q^{r-1}+2-2q^{r-1}]$

$= (pq^{r-1})[2-q^{r-1}-q^r]$ **A1 (GIVEN ANSWER)**

6

Expn. $= \displaystyle\sum_{r=1}^{\infty} r\;p\;q^{r-1}(2-q^{r-1}-q^r)$ **M1**

$= 2p\;(1+2q+3q^2+\ldots) - p(1+2q^2+3q^4+\ldots) - pq(1+2q^2+3q^4+\ldots)$

A1 A1 for correct series identified (one of each kind)

$= 2p\,.\,\dfrac{1}{(1-q)^2} - p(1+q)\,.\,\dfrac{1}{(1-q^2)^2}$ **B1 B1 for correct use of given result**

$= \dfrac{2p}{p^2} - \dfrac{p(1+q)}{p^2(1+q)^2} = \dfrac{2(1+q)-1}{p(1+q)}$ or $\dfrac{2q+1}{p(1+q)}$ or $\dfrac{3-2p}{p(2-p)}$ **A1**

or any other correct alternative form with p's / q's

6

(ii) **M1 for equating their answer (in terms of *p* only) to *m*:** $m = \dfrac{3-2p}{p(2-p)}$

dM1 for creating a quadratic eqn. in *p*: $0 = m\,p^2 - 2(1 + m)\,p + 3$ **A1 correct**

dM1 for use of the quadratic formula:

$$p = \frac{2(m+1) \pm \sqrt{4(m^2+2m+1)-12m}}{2m} = \frac{1}{m}\left\{m+1 \pm \sqrt{m^2-m+1}\right\}$$

A1 for correct, simplified answer

A1 for choosing correct answer: $p = \dfrac{1}{m}\left\{m+1-\sqrt{m^2-m+1}\right\}$

M1 A1 for explaining reasons for rejecting other answer:

e.g. With the + sign, $p = 1+\dfrac{1}{m}+$ (something positive) > 1

8

13 **M1 for use of** $e^{-x} \approx 1 - x$ **and applying this to** $\frac{n-r}{n} = 1 - \frac{r}{n}$

A1 for getting $\frac{n-r}{n} \approx e^{-r/n}$ **(GIVEN ANSWER)**

2

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p(at least one matching pair)

= 1 – p(no matching pairs) **M1**

$= 1 - \frac{365}{365} \times \frac{364}{365} \times \ ..... \ \times \frac{365-(k-1)}{365}$ **A1**

$= 1 - 1 \times \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \ ..... \ \left(1 - \frac{k-1}{365}\right)$ **M1 for attempt to use above result**

$\approx 1 - e^{-1/365} \times e^{-2/365} \times \ ..... \ \times e^{-(k-1)/365}$ **A1**

$= 1 - \exp\left\{-\frac{1}{365}(1 + 2 + \ ..... \ + [k-1])\right\}$ **M1 for use of appropriate law of indices**

$= 1 - \exp\left\{-\frac{1}{365} \times \frac{k(k-1)}{2}\right\}$ **dM1 for use of AP-sum**

$= 1 - \exp\{-k(k-1)/730\}$ **A1 (GIVEN ANSWER)**

7

Require $1 - \exp\{-k(k-1)/730\} \geq \frac{1}{2}$ **M1**

dM1 for $\exp\{-k(k-1)/730\} \leq \frac{1}{2}$ **and taking logs**

$$-\frac{k(k-1)}{730} \leq -\ln 2$$

$$k^2 - k \geq 730 \times \frac{253}{365}$$ **B1 for use of given approx$^{\text{n}}$. to ln 2**

M1 for solving quadratic inequality in *k*: $k^2 - k - 506 \geq 0$

either by completing the square: $4k^2 - 4k - 2024 \geq 0 \Rightarrow (2k-1)^2 \geq 45^2$

or by factorising: $(k - 23)(k + 22) \geq 0$ (or by the quadratic formula)

A1 for answer $k \geq 23$

5

$$P_H = 1-\left(\frac{N-1}{N}\right)^k = 1-\left(1-\frac{1}{N}\right)^k \approx 1-\left(e^{-1/N}\right)^k = 1-e^{-k/N}$$

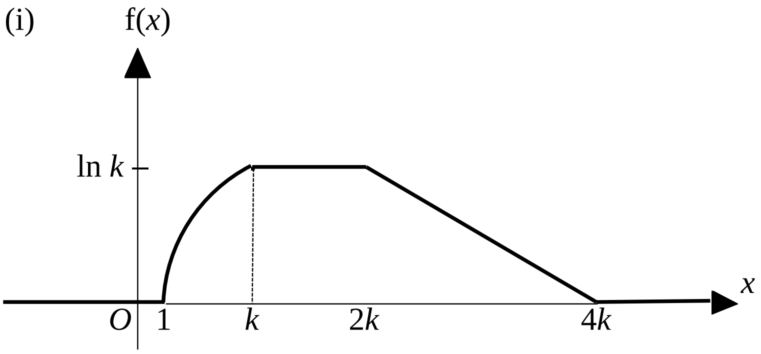
M1 A1 M1 for attempt at expl. form A1

$$P_H \geq \frac{1}{2} \Rightarrow e^{-k/N} \leq \frac{1}{2} \Rightarrow \frac{k}{N} \geq \ln 2 \Rightarrow k \geq N \ln 2 = 253$$

M1 for solving, including taking logs A1

6

14 (i)



M1 for a continuous graph
M1 for 3 (5) pieces

B1 for correct vertices

A1 all correct (else)

4

(ii) $f(2k) = a - 2kb = \ln k$ $\quad$ $f(4k) = a - 4kb = 0$

M1 for subst^n. and solving attempt

$\Rightarrow a = 4kb \Rightarrow b = \dfrac{\ln k}{2k}$ and $a = 2\ln k$ $\quad$ A1 for both answers

Total Prob. $= 1 = \displaystyle\int_1^k \ln x \, dx + k\ln k + \int_{2k}^{4k} (a - bx)\, dx$

M1 $\quad$ M1 for three integrals/areas

$\Rightarrow 1 = \left[x\ln x - x\right]_1^k \quad + \; k\ln k \; + \; \left[ax - \tfrac{1}{2}bx^2\right]_{2k}^{4k}$

M1 A1 by parts

$\Rightarrow 1 = k\ln k - k + 1 \quad + \; k\ln k \; + \; 2ak - \tfrac{1}{2}b \,.\, 12k^2$

A1 for both of these

M1 for subst^n. of limits and use of their a and b in terms of k

$\Rightarrow 0 = k\ln k - k + k\ln k + \; 4k\ln k - 3k\ln k$

$k \neq 0 \Rightarrow k = e^{1/3}$ and so $a = \dfrac{2}{3}$, $b = \dfrac{1}{6}e^{-1/3}$

M1 for obtaining numerical answers for k, a and b
A1 for all correct

10

(iii) **B1 for** $\int_1^k \ln x \, dx = k \ln k - k + 1$ (from earlier) $= 1 - \frac{2}{3} e^{1/3}$

and $1 - \frac{2}{3} e^{1/3} < \frac{1}{2} \Leftarrow e^{1/3} > \frac{3}{4}$ which it is since $e^{1/3} > 1$

B1 for showing median not in first region

B1 for $\int_1^k \ln x \, dx + k \ln k = 1 - \frac{1}{3} e^{1/3}$

and $1 - \frac{1}{3} e^{1/3} > \frac{1}{2} \Leftarrow e^{1/3} < \frac{3}{2} \Leftarrow e < \frac{27}{8}$ which it is since $e < 3$

B1 for showing median *is* in second region

Then median is given by $1 - \frac{2}{3} e^{1/3} + (m - k) \ln k = \frac{1}{2}$ **M1**

$\Rightarrow m = 3 e^{1/3} - \frac{3}{2}$ **A1**

6